Some observers designs for finite dimensional nonlinear systems

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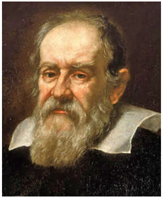
Work (mainly) in collaboration with P. Bernard and L. Praly

Approaching the observation problem

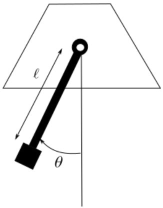
Measure what is measurable and make it measurable what is not so,
Galileo Galilei, 1564-1642, Italy.



An example of Observation problem

Galileo Galelei wrote in 1638 in a book entitled *Dialogues concerning two new sciences*,

I can easily measure the length of a . . . string whose upper end is attached . . . . For if I attach to the lower end of this string a rather heavy weight and give it a to-and-from motion,...



Hidden variables

$\Rightarrow$ **Estimate the length of a string**

...and if I ask a friend to count a number of its vibration,...

Measurements

$\Rightarrow$ **angular position of a pendulum**

Question : Knowing the angular position of the pendulum θ, can we estimate the length ℓ ?

Galileo Galelei gives an Algorithm to solve this problem.

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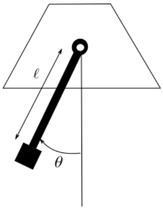

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Let's try to find another one following a model based approach !

The model of the system and output

We consider a model described by a differential equation in the form of

(finite dimensional) nonlinear autonomous model with output

$$\dot{x} = f(x)\ ,\ \ y = h(x)\ ,$$

- x the state is in $\mathcal{X} \subset \mathbb{R}^n$, a positively invariant open subset.
- y the measured output in $\mathbb{R}$
- $t \mapsto X(x_0, t)$ is the solution assumed to be well and uniquely defined
- f and h are sufficiently smooth
- there is no uncertainty = perfect model

We want to estimate $X(x, t)$ from the knowledge of $t \mapsto y(t)$

Example of the length estimation problem

In the case of the length estimation, the model is:

$$\left\{\begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \sin(x_1) \\ \dot{x}_3 = 0 \end{array}\right. \quad , \; y = h(x) = x_1 \; .$$

The system with state $x = (x_1, x_2, x_3)$ in $\mathcal{X} = S^1 \times \mathbb{R} \times \mathbb{R}_+$

The state space is a manifold

We want to estimate x_3 from the knowledge of $t \mapsto y(t) = X_1(x, t)$

An asymptotic observer

An asymptotic state observer

It is a system

$$\hat{x} = \tau(\hat{z}, y) \ , \ \dot{\hat{z}} = \varphi(\hat{z}, y)$$

with state $\hat{z}$ in $\mathcal{Z} \subset \mathbb{R}^m$. such that the following two properties hold:

- For all $(x, \hat{z}) \in \mathcal{X} \times \mathcal{Z}$, the solution $t \mapsto Z(x, \hat{z}, t)$ is well and uniquely defined in positive time.
- For any x in $\mathcal{X}$ and for any $\hat{z}$ in $\mathcal{Z}$:

$$\lim_{t \to +\infty} |X(x, t) - \hat{X}(x, \hat{z}, t)| = 0$$

where $\hat{X}(x, \hat{z}, t) = \tau(Z(x, \hat{z}, t), h(X(x, t)))$.

$\Rightarrow$ Asymptotic convergence of the estimate toward the state

Linear asymptotic observer

For linear systems, the observation problem was solved by D.G. Luenberger in

Observing the State of a Linear System

DAVID G. LUENBERGER, STUDENT MEMBER, IEEE

Summary—In much of modern control theory designs are based on the assumption that the state vector of the system to be controlled is available for measurement. In many practical situations only a few output quantities are available. Application of theories which assume that the state vector is known is severely limited in these cases. In

techniques have been developed to find the function F for special classes of control problems. These techniques include dynamic programming [8]–[10], Pontryagin's maximum principle [11], and methods based on Lya-

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Let's go back to his approach...

Luenberger Approach

Instead of requiring that the observer reconstruct the state vector itself, require only that it reconstruct some constant linear transformation of the state vector. This

Luenberger idea 1 : Reconstruct a function of the state $\tau^*(x)$

Construct

1. a vector field $\varphi(\hat{z}, y)$ on $\mathcal{Z} \subseteq \mathbb{R}^m$, $m \in \mathbb{N}$
2. a mapping $\tau^* : \mathcal{X} \to \mathcal{Z}$

such that along the system $\dot{x} = f(x)$, $\dot{\hat{z}} = \varphi(\hat{z}, h(x))$ the set $\{\hat{z} = \tau^*(x)\}$ is invariant and attractive, i.e.

$$\lim_{t \to +\infty} |Z(\hat{z}, x, t) - \tau^*(X(x, t))| = 0$$

$\Rightarrow$ *We asymptotically estimate $\tau^*(x)$.*

Luenberger Approach

III. OBSERVATION OF THE ENTIRE STATE VECTOR

related by a constant linear transformation. The question which naturally arises now is: How does one guarantee that the transformation obtained will be invertible?

Luenberger idea 2 : If $x \mapsto (\tau^*(x), h(x))$ has a left inverse we have a state observer.

Luenberger Approach

In conclusion :

Observer design approach

Design a vector field $\varphi(\hat{z}, y)$ and a mapping $\hat{z} = \tau^*(x)$ such that

- along the system the set $\{\hat{z} = \tau^*(x)\}$ is an invariant and attractive manifold
- τ^* is injective relatively to the output i.e. there exists a mapping τ such that $\tau(\tau^*(x), h(x)) = x$

The obtained observer is

$$\dot{\hat{z}} = \varphi(\hat{z}, y) \ , \ \hat{x} = \tau(\hat{z}, y)$$

Outlines of the presentation

In this talk, following this route, we give 3 different approaches:

1. Observers based on detectability conditions
2. Observers based on observability assumptions
    1. High-gain observers
    2. Nonlinear Luenberger observers
3. The left inversion problem

An (almost) necessary condition

Theorem (Andrieu HDR-2019)

Assume $\mathcal{X}$ is bounded and assume there exists a smooth vector field $\varphi(\hat{z}, y)$ and a smooth mapping $\hat{z} = \tau^*(x)$ such that

- Along the system, the set $\{\hat{z} = \tau^*(x)\}$ is exponentially attractive + stable
- $x \mapsto \begin{bmatrix} \tau^*(x) \\ h(x) \end{bmatrix}$ is injective and full rank.

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Then there exists $P : \mathcal{X} \to \mathbb{R}^{n\times n}$ such that for all (x, v) in $\mathcal{X} \times \mathbb{R}^n$

$$c_1 I \leq P(x) \leq c_2 I$$

$$\frac{\partial h}{\partial x}(x)v = 0 \Rightarrow v' \underbrace{\left(\mathfrak{L}_f P(x) + 2P(x)\frac{\partial f}{\partial x}(x)\right)}_{=L_f P(x)} v \leq -\lambda v' P(x) v$$

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We call this property : infinitesimal detectability.

Sufficient condition based on detectability

Question : Can we get an observer from infinitesimal detectability ?

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Yes, a local one !

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Yes, a local one !

Proposition (Sanfelice-Praly TAC-2017)

Assume the system is infinitesimally detectable + smoothness and bounds on derivatives then, there exists $\underline{k}$ such that for $k \geq \underline{k}$,

$$\dot{\hat{z}} = f(\hat{z}) - k\, P(\hat{z})^{-1} \frac{\partial h}{\partial x}(\hat{z})^\top [h(\hat{z}) - y] \quad , \qquad \hat{x} = \hat{z}$$

is a LOCAL exponential observer.

Sufficient condition based on detectability

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- Semi-global result may be obtained assuming a strong assumption on P and h using the Riemanian length as associated error Lyapunov function.
- Still ongoing research...

Sufficient condition based on detectability

A particular case

Euclidean Infinitesimal Detectability

Assume $h(x) = Hx$ (linear in x) and (f, h) is infinitesimally detectable with P constant (Euclidean metric), i.e, for all (x, v) in $\mathbb{R}^n \times \mathbb{R}^n$

$$Hv = 0 \;\Rightarrow\; v^T P \frac{\partial f}{\partial x}(x)\, v \;\leq\; -\lambda\, v^T P\, v$$

Sufficient condition based on detectability

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Proposition (Praly IEEE-TAC-2001)

Then, for for each compact subset E of $\mathbb{R}^n$, we can find a function ℓ_E such that the observer

$$\dot{\hat{z}} \ = \ f(\hat{z}) \ - \ \ell_E(\hat{z})\, P^{-1} H^T [H\hat{z} - y] \quad , \qquad \hat{x} \ = \ \hat{z}$$

solves the observer convergence problem for all initial condition $(x, \hat{z})$ in $\mathbb{R}^n \times \mathbb{R}^n$ satisfying $x - \hat{z} \in E$.

$\Rightarrow$ This is a semi-global result.

Case of the length estimation problem

To apply the previous approach, we need to find

$$P(x) = \begin{bmatrix} P_{11}(x) & P_{12}(x) & P_{13}(x) \\ P_{12}(x) & P_{22}(x) & P_{23}(x) \\ P_{13}(x) & P_{23}(x) & P_{33}(x) \end{bmatrix}$$

such that $P(x) > 0$ and

$$\begin{bmatrix} 0 & v_2 & v_3 \end{bmatrix} \left[\frac{\partial P}{\partial x_1}(x) x_2 + \frac{\partial P}{\partial x_2}(x) x_3 \sin(x_1) \right] \begin{bmatrix} 0 \\ v_2 \\ v_3 \end{bmatrix}$$

$$+ 2 \begin{bmatrix} 0 & v_2 & v_3 \end{bmatrix} P(x) \begin{bmatrix} 0 & 1 & 0 \\ x_3 \cos(x_1) & 0 & \sin(x_1) \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ v_2 \\ v_3 \end{bmatrix} < 0 \quad \forall (v_2, v_3)$$

$\Rightarrow$ 5 inequalities have to be satisfied by the 6 functions P_{ij}.
An explicit solution is given by L. Praly for x in $(-\pi, \pi) \times \mathbb{R} \times \mathbb{R}_+$.

Sufficient condition based on detectability

From Euclidean Infinitesimal detectability to contraction

Euclidean Infinitesimal Detectability

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Finsler like relaxation

There exist P, $\ell > 0$ and $\lambda > 0$ such that

$$P \frac{\partial f}{\partial x}(x) + \frac{\partial f}{\partial x}(x)^\top P - 2\ell H^\top H < -\lambda P$$

Sufficient condition based on detectability

From Euclidean Infinitesimal detectability to contraction

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In that case,

$$\dot{\hat{z}} = f(\hat{z}) - \ell P^{-1} H^\top (H\hat{z} - y)$$

is a (uniform in y) CONTRACTION and hence, a GLOBAL observer.

Remark: Finding (P, ℓ) requires to solve an infinite dimensional LMI which may not have a solution.

Observers from infinitesimal detectability

By restricting ourselves to some particular class of vector fields, some finite dimensional LMI may be obtained

- Global lipschitz and structured vector field (the high-gain approach)
- Lipschitz vector fields in some directions (see the work of Zemouche, ...)
- Slope restricted nonlinearities (see the work of Arcak, Kokotovic, ...)
- Output dependent nonlinearities

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A solution (highly) investigated in the literature is to find a diffeomorphism which leads to these properties. (See the work of Respondek, Nijmeier, Krener, Isidori, Hammouri . . .)

Outlines of the presentation

1. Observers based on detectability conditions
2. Observers based on observability assumptions
    1. Nonlinear Luenberger observers
    2. High-gain observers
3. The left inversion problem

Nonlinear Luenberger observers

Following Luenberger approach, we consider the particular case in which

$$\dot{\hat{z}} = \varphi(\hat{z}, y) = A\hat{z} + B(y)$$

with A in $\mathbb{C}^{m\times m}$ Hurwitz.

Following the observer design receipe

We wish to design a mapping $\tau^*(x)$ such that

1. along the system the set $\{\hat{z} = \tau^*(x)\}$ is an invariant manifold and attractive
2. τ^* is injective relatively to the output i.e. there exists a mapping τ such that $\tau(\tau^*(x), h(x)) = x$

This approach for nonlinear system has been initiated by Shoshitaishvili (Theo. of Sing. and its applications, 1991) and Kazantzis-Kravaris (SCL-1998)

Nonlinear Luenberger Observer Approach

For the system

$$\dot{x} = f(x) \ , \ \dot{\hat{z}} = A\hat{z} + B(h(x))$$

If τ^* is solution of

$$\frac{\partial \tau^*}{\partial x}(x) f(x) \ = \ A\tau^*(x) \ + \ B(h(x))$$

then $\hat{z} = \tau^*(x)$ is trivially invariant and attractive

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So, two questions :

- When does a solution, τ^* exist ? How does it looks like ?
- When is this solution τ^* injective ?

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To simplify presentation we assume that $\mathcal{X}$ is backward complete

Question 1 : When does τ^* exists ?

Always !

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Always !

Theorem (Andrieu-Praly (SICON 06), see also Kreisselmeir-Engel (IEEE TAC 03))

For each Hurwitz matrix A we can find a C^1 function B such that a weak solution τ^* exists.
Moreover, if h is C^1, there exists ℓ such that τ^* is C^1 if

$$real(eigen(A)) \leq -\ell$$

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A General Expression :

$$\tau^*(x) \ = \ \int_{-\infty}^{0} \exp(-As)\, B(h(X(x,s)))\, ds$$

How does it look like on example ?

For the length estimation problem

In the case of the length estimation, the model is:

$$\left\{ \begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \sin(x_1) \\ \dot{x}_3 = 0 \end{array} \right. \quad , \ y = h(x) = x_1 \ .$$

We take A diagonal. For each eigen value λ of A, the pde is:

$$\frac{\partial {\tau^*}_\lambda}{\partial x_1}(x)\, x_2 \ + \ \frac{\partial {\tau^*}_\lambda}{\partial x_2}(x)\, x_3 \sin(x_1) \ = \ \lambda\, {\tau^*}_\lambda(x) \ + \ \sin(x_1)$$

a solution is $\qquad {\tau^*}_\lambda(x) =$ Mapple... too complex !

Trick = Dynamic extension

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We take A diagonal. For each eigen value λ of A, the pde is:

$$\frac{\partial {\tau^*}_\lambda}{\partial x_1}(x)\, x_2 \; + \; \frac{\partial {\tau^*}_\lambda}{\partial x_2}(x)\, x_3 \sin(x_1) + \frac{\partial {\tau^*}_\lambda}{\partial c}(x)\, \varphi(x_1, c) \; = \; \lambda\,({\tau^*}_\lambda(x) \; + \; \sin(x_1))$$

There exists solution in the form

$$\tau^*(x) = c_1 x_3 + c_2 x_2 + \sin(x_1)$$

where,

$$\dot{c}_1 = \varphi_1(x_1, c) = \lambda c_1 + c_2 \sin(x_1) \; , \; \dot{c}_2 = \varphi_2(x_1, c) = \lambda c_2 + \cos(x_1)$$

Question 2 : When is this solution injective ?

Depends on observability and m!

1. Backward distinguishability
2. On the length estimation problem

Backward distinguishability

Backward distinguishability

$\forall\ x_1 \neq x_2$ in $\mathtt{cl}(\mathcal{X})$ there exists a negative time t, such that we have :

$$h(X(x_1,t)) \neq h(X(x_2,t)) \ .$$

$=$ *The present state x can be distinguished from other states by looking at the past output paths*

Question 2 : When is this solution injective ?

Theorem (Andrieu-Praly (SIAM J. C. and Opt.-2006))

Assume the system is backward distinguishable. Then, with choosing any injective C^1 function $b : \mathbb{R}^p \to \mathbb{C}$, and picking $m = n + 1$, $\exists$ a positive real number ℓ and a subset S of $\mathbb{C}^{n+1}$ with zero Lebesgue measure such that the function $\tau^* = ({\tau^*}_{\lambda_i}) : cl(\mathcal{X}) \to \mathbb{C}^{(n+1)}$ defined as :

$$ {\tau^*}_{\lambda_i}(x) \; = \; \int_{-\infty}^{0} \exp(-\lambda_i s) \, b(h(X_m(x, s))) \, ds $$

is injective provided the $n + 1$ complex numbers λ_i are (arbitrary) in $\mathbb{C}^{n+1} \setminus S$ and with real part strictly smaller than ℓ.

$\Rightarrow$ If the system is Backward distinguishable, there exists an observer !

Question 2 : When is this solution injective ?

case of the length estimation problem

We have

$$\tau^*(x) = M(t) \begin{bmatrix} x_3 \\ x_2 \end{bmatrix} + B_\lambda \sin(x_1)$$

where

$$M(t) = \begin{bmatrix} c_{1,1} & c_{1,2} \\ \vdots & \vdots \\ c_{m,1} & c_{m,2} \end{bmatrix}$$

where,

$$\dot{c}_{i,1} = \lambda c_{i,1} + c_{i,2} \sin(x_1) \ , \ \dot{c}_{i,2} = \lambda c_{i,2} + \cos(x_1)$$

It can be shown that M is of rank 2 if we pick $m = 3$

Case of the length estimation problem

Ultimately the nonlinear Luenberger observer is, for $i \in \{1,2,3\}$,

$$\dot{\hat{z}}_i = \lambda_i \left(\hat{z}_i - \sin(x_1)\right)$$

$$\dot{c}_{i1} = \lambda_i \, c_{i1} + c_{i2} \sin(x_1) \quad , \qquad \dot{c}_{i2} = \lambda_i \, c_{i2} - \cos(x_1)$$

$$M_3 = \begin{pmatrix} c_{1\ell} & c_{1\omega} \\ c_{2\ell} & c_{2\omega} \\ c_{3\ell} & c_{3\omega} \end{pmatrix}$$

$$\hat{x}_3 = \frac{1}{(1 \quad 0)\left(M_3^T M_3\right)^{-1} M_3^T \left[\hat{z} - \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \sin(x_1)\right]}$$

Conclusion on the Nonlinear Luenberger observer

- A Nonlinear Luenberger observer is theoritically a very powerfull tool. If the system is observable, there exists one !
- Need to solve a PDE
- Has been employed with success for the estimation of mechanical coordinates in electrical machines[1]
- It is a very powerfull tool for identification of linear systems[2]
- Can be employed for controlled nonlinear system[3]

[1] Bernard-Praly TAC-2019
[2] Afri-Andrieu-Bako-Dufour TAC-2016
[3] Bernard-Andrieu TAC-2019

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Tunnable observers

Consider

- the autonomous system
- $\mathfrak{O}$ a "set" of observers
- $\mathcal{A} \subset \mathcal{X}$ a bounded set of initial conditions

Tunnable observer set

For all time t_d, for all $\epsilon > 0$ there exists an asymptotic observer in $\mathfrak{O}$

$$\dot{\hat{z}} = \varphi(\hat{z}, y) \ , \ \hat{x} = \tau(\hat{z}, y)$$

such that for all x we have

$$\left| X(x,t) - \hat{X}(0,x,t) \right| \leq \epsilon \ , \ \forall \ t \geq t_d \ , \ \forall x \in \mathcal{A}$$

Necessary condition for tunable observers

Proposition (Andrieu et.al IEEE CDC 2014)

Assume there exists a set $\mathfrak{O}$ of tunable observer. Then for all x_a and x_b in $\mathcal{A}^2$ such that there exists t_d such that

$$h(X(x_a,t)) = h(X(x_b,t)) \ , \ \forall \ t \in [0, t_d)$$

then we have,

$$x_a = x_b \ .$$

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then we have,

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This property is called Instantaneous distinguishability

When the system is sufficiently smooth, this fast property may be checked from the time derivatives of the output.

On the length estimation problem

In the case of the length estimation, the model is:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \sin(x_1) \\ \dot{x}_3 = 0 \end{cases} \quad , \; y = h(x) = x_1 \; .$$

Consider

$$H_4(x) = \begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \\ y^{(3)} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \sin(x_1) \\ x_3 x_2 \cos(x_1) \end{bmatrix}$$

$\Rightarrow$ injective in an open subset $\mathcal{A}$ of $\mathcal{X}$ defined as

$$\mathcal{A} = \{x \in \mathcal{X}, (x_1, x_2) \neq 0\}$$

$\Rightarrow$ the mapping H_4 is injective *at least* in $\mathcal{A}$ and the pendulum model is instantaneously distinguishable at $\mathcal{A}$.

On the length estimation problem

1. By looking at the successive output time derivatives evaluated at time 0 we may get an injective map on some open sets of the state space.

[4] See Jouan-Gauthier (JDCS 1996), Andrieu (SIAM J. C. and Opt.-2014)

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These facts are true for all analytic systems[4].

[4] See Jouan-Gauthier (JDCS 1996), Andrieu (SIAM J. C. and Opt.-2014)

In Conclusion

It gives some hint on how to solve the estimation problem:

$$y(t)$$

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Estimate the $m-1$ first time derivatives of the output at each time:

$$\tau^*(x) = \widehat{H_m(x)} = \left(\widehat{y}\ ,\ \widehat{\dot{y}}\ ,\ \widehat{\ddot{y}}\ \ldots\ ,\ \widehat{y^{m-1}}\right)$$

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Apply τ such that $\tau(H_m(x)) = x$

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This is the high-gain observer approach

Model of the output derivatives

Assumption

There exists $m \geq 0$ and an open subset $\mathcal{A}$ of $\mathcal{X} \subset \mathbb{R}^n$ such that the mapping $H_m : \mathcal{A} \to H_m(\mathcal{A}) \subset \mathbb{R}^m$ is injective.

Along the solution of the system we have,

$$\dot{\overbrace{H_m(x)}} = A\, H_m(x) + BL_f h_m(x)$$

with

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots \\ \vdots & & \ddots & \\ & & & 1 \\ \dots & & & 0 \end{bmatrix} , \; B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} .$$

Model of the output derivatives

Injectivity of $H_m \Rightarrow$ there exists $\varphi_m : \mathbb{R}^m \to \mathbb{R}^m$ such that

$$\varphi_m(H_m(x)) = L_f h_m(x) \ , \forall \ x \in \mathcal{A} \ .$$

$\Rightarrow z = H_m(x)$ is an invariant manifold of the system

$$\dot{x} = f(x) \ , \ \dot{z} = Az + B\varphi_m(z) \ ,$$

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$$\dot{x} = f(x) \ , \ \dot{z} = Az + B\varphi_m(z) \ ,$$

1. We can focus on estimating the state of the z system with $y = z_1$
2. Due to the structure of the z subsystem, we know how to design P solution to the (Euclidean) detectability equation !

$$v_1 = 0 \Rightarrow v'P\left[A + B\frac{\partial \varphi_m}{\partial z}(z)\right]v < 0$$

Example

The length estimation problem

The model and the immersion are:

$$\left\{\begin{array}{l}\dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \sin(x_1) \\ \dot{x}_3 = 0\end{array}\right. \quad , \; z = \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \sin(x_1) \\ x_3 x_2 \cos(x_1) \end{array}\right]$$

when $\sin(z_1)^2 + z_2^2 \cos(z_1)^2 \neq 0$, it yields

$$\left\{\begin{array}{lcl}\dot{z}_1 & = & z_2 \\ \dot{z}_2 & = & z_3 \\ \dot{z}_3 & = & z_4 \\ \dot{z}_4 & = & f_{z4}(z) = (\tau(z))_3^2\, z_2 \sin(z_1)\cos(z_1) - (\tau(z))_3\, z_2^2 \sin(z_1)\end{array}\right.$$

with,

$$(\tau(z))_3 = \frac{z_3 \sin(z_1) + z_4 z_2 \cos(z_1)}{\sin(z_1)^2 + z_2^2 \cos(z_1)^2} \; .$$

Knowing $y = \hat{z}_1$ can we design an observer for this system ? YES !!

High-gain approach

The model of the output derivative can be rewritten :

$$\underbrace{\dot{z} = Az}_{\text{Chain of integrator part}} \quad + \quad \underbrace{\varphi(z)}_{\substack{\text{Triangular Nonlinearities} \\ = \text{Disturbances}}}$$

HIGH GAIN IDEA: Consider the nonlinear terms as disturbances

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HIGH GAIN IDEA: Consider the nonlinear terms as disturbances

Two steps for the design of the observer

1. Synthesize a robust observer for a linear system.
2. Amplify the robustness to deal with φ.

High-gain approach

Several contexts can be considered

- The (well-known) lower triangular Lipschitz context[5]

$$|\varphi_i(z) - \varphi_i(\hat{z})| \leq \Gamma \sum_{j=1}^{i} |z_j - \hat{z}_j|$$

STEP 1 Pick K such that $A + KC$ is Huwitz

STEP 2 Increase the robustness with high-gain scaling

$$\dot{\hat{z}} = A\hat{z} + \varphi(\hat{z}) + \begin{bmatrix} L & & \\ & \ddots & \\ & & L^n \end{bmatrix} K(\hat{z}_1 - y) \ , \ L \geq a\Gamma$$

[5] See Emelyanov et al-89, Tornambe-89, Bornard-Hammouri-91, Gauthier et al-92, Khalil

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Remark : If L increases :

1. Concergence rates increases
2. Picking phenomena increases
3. Sensor robustness decreases

[5] See Emelyanov et al-89, Tornambe-89, Bornard-Hammouri-91, Gauthier et al-92, Khalil

High-gain approach

- The homogeneous contex[6] (with restriction on Γ and $\alpha_{ij} \geq 1$)

$$|\varphi_i(z,y) - \varphi_i(\hat{z},y)| \leq \Gamma(\hat{z},y) \sum_{j=1}^{i} |z_j - \hat{z}_j| + c_\infty \sum_{j=1}^{i} |z_j - \hat{z}_j|^{\alpha_{ij}}$$

STEP 1 Select k a nonlinear corection terms (homogeneous in the bi-limit)[7]

STEP 2 Adapt the convergence with dynamic high-gain scaling

$$\dot{\hat{z}} = A\hat{z} + \varphi(\hat{z}) + \begin{bmatrix} L & & \\ & \ddots & \\ & & L^n \end{bmatrix} k(\hat{z}_1 - y) \ , \ \dot{L} = L(a_1(a_2 - L) + a_3\Gamma(\hat{z},y))$$

Remark : L learns the local Lipschitz constant and doesn't need to be too large.

[6] Andrieu-Praly-Astolfi 2009

[7] From Andrieu-Praly-Astolfi, Homogeneous approximation, recursive observer design and output feedback, 2008

- The bounded context[8]

$$|\varphi_j(z)| \leq \varphi_{\max} \;\;\&\;\; |z_i| \leq z_{\max}$$

STEP 1 Select homogeneous set-valued corection terms
STEP 2 Use of cascaded finite time observers

$$\dot{\hat{z}}_i \in A_i \hat{z}_i + \varphi_i(\hat{z}_{i-1}) + k_i(\hat{z}_1 - y)$$

[8] Floquet-Barbot-2007, Bernard-Praly-Andrieu-2-2017
[9] Bernard-Praly-Andrieu-1-2017

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Remark 1: This context is important since for uniformly observable systems with input

$$\dot{x} = f(x) + g(x)u \ , \ \ y = h(x)$$
$$\Downarrow$$
$$\dot{z} = A\, z + \varphi(z, u) \ , \ \ y = z_1$$

with φ (possibly) triangular but generally only continuous[9].

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$$\dot{z} = A\,z + \varphi(z,u) \;, \;\; y = z_1$$

with φ (possibly) triangular but generally only continuous[9].

Remark 2: It is an open question to characterize which (controlled) nonlinear system can be immersed into a Lipschitz triangular form.

[8] Floquet-Barbot-2007, Bernard-Praly-Andrieu-2-2017
[9] Bernard-Praly-Andrieu-1-2017

Example
The length estimation problem

Ultimately, the high-gain observer for the length estimation problem is

$$\left\{\begin{array}{rcl} \dot{\hat{z}}_1 & = & \hat{z}_2 + L k_1 (y - \hat{z}_1) \\ \dot{\hat{z}}_2 & = & \hat{z}_3 + L^2 k_2 (y - \hat{z}_1) \\ \dot{\hat{z}}_3 & = & \hat{z}_4 + L^3 k_3 (y - \hat{z}_1) \\ \dot{\hat{z}}_4 & = & (\tau(\hat{z}))_3^2 \, \hat{z}_2 \sin(y)\cos(y) - (\tau(\hat{z}))_3 \, \hat{z}_2^2 \sin(y) + L^4 k_4 (y - \hat{z}_1) \end{array}\right.$$

with,

$$(\tau(\hat{z}))_3 = \frac{\hat{z}_3 \sin(y) + \hat{z}_4 \hat{z}_2 \cos(y)}{\sin(y)^2 + \hat{z}_2^2 \cos(y)^2} \ .$$

and,

$$\hat{x}_1 = \hat{z}_1 \ , \ \hat{x}_2 = \hat{z}_2 \ , \ \hat{x}_3 = (\tau(\hat{z}))_3$$

Conclusion on the high-gain observer

- High-gain observer are the most popular observer
- It has some issues.
- Somme tools may be employed to attenuate them
  - The use of cascade of second order observers
    - Cascaded homogeneous observers (see the work of J.-P. Barbort)
    - Cascaded high-gain observers (see the work of D. Astolfi)
  - Use of hybrid tools (see the work of Zaccarian, Tarbouriech, Astolfi...)

Outlines of the presentation

1. Observers based on detectability conditions
2. Observers based on observability assumptions
    1. Nonlinear Luenberger observers
    2. High-gain observers
3. The left inversion problem

The left inversion problem for observer

The receipe :

Observer design approach

Design a vector field $\varphi(\hat{z}, y)$ and a mapping $\hat{z} = \tau^*(x)$ such that

- along the system the set $\{\hat{z} = \tau^*(x)\}$ is an invariant and attractive manifold
- τ^* is injective relatively to the output i.e. there exists a mapping τ such that $\tau(\tau^*(x), h(x)) = x$

The obtained observer is

$$\dot{\hat{z}} = \varphi(\hat{z}, y) \ , \ \hat{x} = \tau(\hat{z}, y)$$

To construct the observer, we need to construct τ such that $\tau(\tau^*(x), h(x)) = x$.

The left inversion problem for observer

- The construction of a left inverse to τ^* may be very difficult
- Most of the time rely on the use of optimization procedure. A possible solution for τ is given by:

$$\tau(z) = \mathtt{ArgMin}_x \left|\tau^*(x, h(x)) - z\right|^2$$

  Problem : This approach, based on optimization procedure may be difficult to implement !

Given an observer, can we *realize* it without computing the left inverse?

The diffeomorphism case

In the particular case where τ^* is a diffeomorphism $\tau^* : \mathcal{X} \subset \mathbb{R}^n \to \tau^*(\mathbb{R}^n) = \mathbb{R}^n$. The observer may be implemented in the original coordinates as

$$\dot{\hat{x}} = \left(\frac{\partial \tau^*}{\partial x}(\hat{x})\right)^{-1} \varphi(\tau^*(\hat{x}), y)$$

$\Rightarrow$ In this case we don't need the left inverse !

Can we extend this approach to the case where τ^* is an immersion $\tau^* : \mathcal{X} \subset \mathbb{R}^n \to \tau^*(\mathbb{R}^n) \subset \mathbb{R}^m$?

A possible solution without inversion

Theorem (Bernard-Andrieu-Praly 2017)

Assume there exists an open subset $\mathcal{O} \subset \mathbb{R}^m$ containing $\mathcal{X} \times \{0\}$ and a diffeomorphism $\tau_e^* : \mathcal{O} \to \mathbb{R}^m$ satisfying

$$\tau_e^*(x, 0) \ = \ \tau^*(x) \qquad \forall x \in \mathcal{X} \ ,$$

and

$$\tau_e^*(\mathcal{O}) = \mathbb{R}^m$$

Then the system

$$\dot{\overbrace{\begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix}}} = \left(\frac{\partial \tau_e^*}{\partial(\hat{x}, \hat{w})}(\hat{x}, \hat{w}) \right)^{-1} \varphi(\tau_e^*(\hat{x}, \hat{w}), \hat{x}, y)$$

defines an observer, i.e.

$$\lim_{t \to +\infty} \left| \hat{W}(x, \hat{x}, \hat{w}, t) \right| + \left| X(x, t) - \hat{X}(x, \hat{x}, \hat{w}, t) \right| = 0 \ .$$

A possible solution without inversion

With this theorem, we introduce two subproblems

Problem 1: Augmenting an immersion into a diffeomorphism

Given an injective immersion $\tau^* : \mathcal{X} \subset \mathbb{R}^n \to \tau^*(\mathcal{X}) \subset \mathbb{R}^m$ can we find a couple $(\tau_a^*, \mathcal{O})$ such that $\tau_a^* : \mathcal{O} \subset \mathbb{R}^m \to \tau_a^*(\mathcal{O}) \subset \mathbb{R}^m$ is a diffeomorphism, $\mathtt{cl}(\mathcal{X} \times \{0\}) \subset \mathcal{O}$ and

$$\tau_a^*(x,0) \;=\; \tau^*(x) \qquad \forall x \in \mathcal{X} \;,$$

Problem 2: Image extension of a diffeomorphism

Let $\mathcal{O}$ be an open subset of $\mathbb{R}^m$ and τ_a^*: $\mathcal{O} \to \tau_a^*(\mathcal{O}) \subset \mathbb{R}^m$ be a diffeomorphism. Given a compact set K in $\mathcal{O}$, can we find a diffeomorphism $\tau_e^* : \mathcal{O} \to \mathbb{R}^m$ such that

$$\tau_e^*(\mathcal{O}) = \mathbb{R}^m \quad , \qquad \tau_e^*(x) = \tau_a^*(x) \quad \forall x \in K.$$

A possible solution without inversion

Proposition (Bernard-Andrieu-Praly 2017)

If $\mathcal{X}$ is contractible, then Problem 1 is solvable. Moreover for some values of m and n we can give explicit solution for τ_a^*.

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If $\mathcal{O}$ is C^2-diffeomorphic to $\mathbb{R}^m$, then Problem 2 is solvable.

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Proposition (Bernard-Andrieu-Praly 2017)

If $\mathcal{O}$ is C^2-diffeomorphic to $\mathbb{R}^m$, then Problem 2 is solvable.

By restricting ourselves to some specific data and by using some hybrid tools some constructive methode are possible (see P. Bernard-L. Marconi 2019).

Conclusion

- Observer for finite dimensional nonlinear systems is still an active research area
- Some problems that need to be adressed:
    - Robustness issue : how to parameterized the observer with respect to the model of uncertainties ? Optimal observer ?
    - Problem of system with inputs[10]
    - Global observer from a non Euclidean infinitesimal detecability property
    - Observer as an optimization algorithm (see the work of Possieri-Menini)
    - Observer for PDEs, for systems with delays...

[10] Bernard-Praly-Andrieu Automatica 2017

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THANK YOU !

[10] Bernard-Praly-Andrieu Automatica 2017